Introduction to Stochastic Processes
EXAMINATION- first session
Year : 2013 - M1 TSE
Course : O. Faugeras

1 hour 30 min. Exercises are independent. No documents authorised.
It is advised to provide careful reasoning and justifications in your answers. It will be taken a great care of them in the notation.

Exercise 1 Conditional expectation
Let X, Y be two $\mathcal{N}(0,1)$ independent random variables. Set $Z = X + Y$. Compute $E(Z|X)$ and $E(X|Z)$.

Exercise 2 Brownian Motion
Let $B := (B_t, t \geq 0)$ a Wiener process defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$ the canonical filtration generated by B.

1. Show that $X := (X_t = B_t^2 - t, t \geq 0)$ is a Brownian motion.
2. Let $a < 0 < b$, $T = \inf\{t \geq 0, B_t = a \text{ or } B_t = b\}$.
   (a) Show that $E(X_{T \wedge t}) = 0$, for all $t \in \mathbb{R}^+$.
   (b) Deduce that $E[T \wedge t] \leq (|a| + b)^2$, for all $t \in \mathbb{R}^+$.
   (c) Deduce that $E(T) < \infty$.
   (d) Deduce from the previous question that $EB_T = 0$. (hint : use Doob's optional stopping theorem).
3. Set $s > 0$ fixed. Show that $\tilde{B} := (\tilde{B}_t = B_{t+s} - B_s, t \geq 0)$ is a BM, independent of $\mathcal{F}_s$.

Exercise 3 Markov Chains
Imagine you enter a casino with a fortune of i euros and gamble, 1 euro at a time, with probability p of winning 1 euro, and probability $q = 1 - p$ of losing it. The resources of the casino are considered infinite, so there are no upper bounds on your fortune. What is the probability that you leave broke ?

1. Show that your fortune follows a Homogeneous Markov chain. Compute the transition probabilities and draw the corresponding state diagram.
2. Set $h_i = P(\text{leaving the casino broke}|\text{starting from state } i)$ Find the system of equations satisfied by h_i.
3. Solve this system if $p = q$. How do you interpret this result in terms of martingales and fair games ?
4. Solve this system if $0 < p < q < 1$. hint : The general solution to these kinds of recurrence systems is of the form $h_i = A + B(q/p)^i$, with A, B constants to determine.
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