TOULOUSE SCHOOL OF ECONOMICS, 2012-2013
Macroeconomics M1 – Franck Portier
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EXAM

Note: each problem represents 25% of the total grade

PROBLEM I

Consider the problem of choosing an infinite sequence of consumption and capital $\{c_t, k_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^{t-1} \log(c_t),$$

with $\beta \in (0,1)$ being an intertemporal discount factor, subject to an initial value k_0 and a transition equation for capital (budget constraint)

$$k_{t+1} = Ak_t^{\alpha} - c_t, \tag{1}$$

where $A > 0$ and $\alpha \in (0,1)$ are parameters of a production function which uses capital at time t to produce Ak_t^{α} of a good that can be consumed or used as capital next period.

1 – What are the control and the state variables ? Set up the Bellman equation for the recursive version of the problem.

2 – Assume that the value function has the form $V(k_t) = a + b \log k_t$ where a and b are two real coefficients to be determined. Under the assumed value function, find the optimal consumption c_t as a function of k_t. Check that the value function is indeed of the assumed form and find a and b.

3 – Write down the optimal law of motion of capital $k_{t+1} = f(k_t)$. What is the steady state of capital $\bar{k}$? Show that the economy converges to the steady state for any initial capital k_0 (that is, $\lim_{t\to\infty} k_t = \bar{k} \; \forall k_0$). What is the steady state of consumption $\bar{c}$?

4 – What happens to consumption when the level of capital is above the steady state ? (Study the sign of $(c_{t+1} - c_t)/c_t$).

PROBLEM II

Consider the problem of choosing a path for consumption and capital $c(t)$, $k(t)$ to maximize

$$\int_0^{\infty} e^{-\rho t} \log c(t) \mathrm{dt},$$

with $\rho > 0$ being an instantaneous discount factor, subject to an initial value $k(0)$ and a transition equation for capital (budget constraint) given by

$$\dot{k}(t) = Ak(t)^{\alpha} - k(t) - c(t), \tag{2}$$

where $A > 0$ and $\alpha \in (0,1)$ are parameters of an instantaneous production function which uses capital held at time t to produce $Ak(t)^{\alpha}$ of a good that can be consumed or used as capital the next instant.

1 – Show that equation (2) is just the continuous version of equation (1). Write down the continuous time Hamiltonian associated with the problem.

2 – Use the first order conditions of that problem to find a relation between the instantaneous percentage increase in consumption $\dot{c}(t)/c(t)$ and $k(t)$. This condition should be similar to the discrete time problem one. What is the correspondence between the instantaneous discount rate ρ and the discount factor β of the problem in discrete time ?

3 – Find the conditions for $\dot{c}(t) = 0$ and $\dot{k}(t) = 0$ as a function of $c(t)$ and $k(t)$. Are steady state levels the same than in the discrete time version ?

4 – Draw the phase diagram in the space $c(t)$ and $k(t)$ and show a possible shape of the unique saddle path.

PROBLEM III

Consider a Social Planner that maximizes a representative household utility, that is derived from consumption c_t and labor h_t :

$$E_t \sum_{i=0}^{\infty} \beta^i (c_{t+i} - Bh_{t+i})$$

where $\beta \in (0,1)$ denotes the discount factor, E_t is the expectation operator conditional on the information set available as of time t, B is a positive constant. The Planner faces the following constraints (linear-quadratic production function and labor adjustment costs):

$$\begin{cases} c_t &= y_t \\ y_t &= z_t h_t - \frac{a}{2} h_t^2 - \frac{b}{2}(h_t - h_{t-1})^2 \end{cases}$$

where z_t is a technology shock, a is a positive parameter and $b \geq 0$ is the adjustment cost parameter. We assume that the productivity shock z_t follows an autoregressive process of order one

$$z_t = \rho z_{t-1} + \varepsilon_t$$

where $\rho < 1$ and ε_t is iid with zero mean and unit variance.

1 - Write the Social Planner problem with h_t as the only control variable.

2 - Derive the first order conditions of the problem.

3 - What are the dynamic properties of the model?

4 - Compute the solution, that is, express the choice variable h_t as a function of the state variables h_{t-1} and z_t. (you can use forward substitutions or the method of undetermined coefficients).

5 - Compute the dynamic responses of labor to a positive shock to techonology.

PROBLEM IV

Consider the following dynamic optimization problem:

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t, 1-h_t) = \max \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t - A\frac{h_t^{1+\eta}}{1+\eta}\right)^{1-\sigma}}{1-\sigma}$$

subject to

$$c_t + a_{t+1} = w_t h_t + (1+r) a_t$$

where $\beta \in (0,1)$ is the discount factor, c_t is consumption, h_t represents labor supply, a_t are assets, A, σ and η are positive constants and w_t and r denote the real wage and the return to assets respectively.

1 - Determine the optimal labor supply.

2 - Compute the elasticity of labor supply with respect to the real wage.

3 - What's the effect on labor supply of a change in the real wage profile (permanent and transitory changes)? Explain.