UNIVERSITE TOULOUSE 1 CAPITOLE

MASTER 1 in ECONOMICS
MASTER 1 ECONOMIE ET STATISTIQUE

Dynamic optimization / code : M1S16

Lundi 24 Juin 2013 ~ amphi MB1

B. ALZIARY

↪ durée conseillée pour traiter ce sujet : 1 heure
↪ **ATTENTION** : le nom de la matière et son code doivent être **IMPERATIVEMENT** recopiés sur la copie d'examen

PROBLEM

Consider the following optimal sequence problem :

$$(SP) \quad \left\| \begin{array}{l} \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(x_t - \gamma\, x_{t+1} \right) \\ \\ \text{such that } 0 \leq \gamma\, x_{t+1} \leq x_t \\ \text{given } x_0 > 0 \end{array} \right.$$

We suppose $\gamma > 0$ and $0 < \beta < \gamma$.

1. **Connection between the sequence problem and the functional equation**
For a feasible plan $\underline{x_0} = (x_0, ., x_t, .) \in \Pi(x_0)$, the cost along the plan is

$$u(\underline{x_0}) = \sum_{t=0}^{\infty} \beta^t \left(x_t - \gamma\, x_{t+1} \right).$$

The value fonction of the problem (SP) is $v^*(x_0) = \sup_{\underline{x_0} \in \Pi(x_0)} u(\underline{x_0})$.

(a) Show that the value fonction v^* of the problem (SP) satisfies the functional equation :

$$(FE) \qquad v(x) = \sup_{0 \leq b\, y \leq a\, x} \{(x - \gamma\, y) + \beta v(y)\} \qquad \forall x \in \mathbb{R}^+$$

(b) Show that, if a feasible plan $\underline{x_0}^*$ satisfies $v^*(x_0) = u(\underline{x_0}^*)$ then

$$v^*(x_t^*) = \left(x_t^* - \gamma\, x_{t+1}^* \right) + \beta v^*(x_{t+1}^*).$$

2. **Existence of a continuous solution for the functional equation**
We look for a solution of the functional equation in the following space :

$$\mathcal{E} = \left\{ f : \mathbb{R}^+ \to \mathbb{R}, \text{ s. t.}, \exists c_0, c_1 \in \mathbb{R}\ \forall x \in \mathbb{R}^+\ f(x) = c_1\, x + c_0 \right\},$$

Let us define the operator T on $\mathcal{E}$ by

$$Tf(x) = \sup_{0 \leq \gamma\, y \leq x} \{(x - \gamma\, y) + \beta f(y)\} \qquad \forall x \in \mathbb{R}^+$$

(a) Show that if $f \in \mathcal{E}$ then $Tf \in \mathcal{E}$.
(b) Let $v_0(x) = 0$, and define $v^{n+1}(x) = Tv^n(x)$. Show that the sequence converges in a finite number of iterations. Find a solution of the functional equation (FE). Explain why this solution is the value function of the sequence problem and find the optimal plan.