UNIVERSITE TOULOUSE 1 CAPITOLE

TSE
Toulouse School of Economics
École d'économie de Toulouse

Année Universitaire 2012-2013
SESSION 1

MASTER 1

DYNAMIC OPTIMIZATION
(durée 1h30)

Vendredi 14 janvier 2013 ~ 13h30 –15h00

B. ALZIARY

****

Résumé de cours autorisé

PROOF

Do the proof of the following theorems :

Assumption H 1. The correspondence $\Gamma(x)$ is nonempty for all $x \in X$.

Assumption H 2. For all $x_0 \in X$ and $\underline{x_0} = (x_0, x_1, \cdots, x_t, \cdots) \in \Pi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^n F(x_t, x_{t+1})$ exists (although it may by plus or minus infty).

Theorem
Let X, Γ, F and β satisfy **H 1** *and* **H 2**. *Let $\underline{x_0^*} \in \Pi(x_0)$ be an optimal plan from x_0, then*

$$v^*(x_n^*) = F(x_n^*, x_{n+1}^*) + \beta v^*(x_{n+1}^*) \quad n = 0, 1, 2 \cdots$$

Theorem
Let X, Γ, F and β satisfy **H 1** *and* **H 2**. *Let $\underline{x_0^*} \in \Pi(x_0)$ be a feasible plan from x_0, satisfying*

$$v^*(x_n^*) = F(x_n^*, x_{n+1}^*) + \beta v^*(x_{n+1}^*) \quad n = 0, 1, 2 \cdots$$

and

$$\limsup_{n\to+\infty} \beta^n v^*(x_n^*) \leq 0 \quad \Leftrightarrow \quad (\ \forall \varepsilon > 0,\ \ \exists n_0 \ \ \forall n \geq n_0 \quad \beta^n v^*(x_n^*) \leq \varepsilon\ ).$$

Then $\underline{x_0^*}$ is an optimal plan for the initial state x_0.

PROBLEM

Consider the following optimal sequence problem :

$$(SP) \quad \left\| \begin{array}{l} \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F\left(2x_t - x_{t+1}\right) \\ \\ \text{such that } x_t \leq x_{t+1} \leq 2x_t \\ \text{given } x_0 > 0 \end{array} \right.$$

We suppose $0 < \beta < \frac{1}{2}$ and $F \ : \ \mathbb{R}_+ \to \mathbb{R}$ satisfies (HF) :
(HF) F is continuous and satisfies

$$\exists M_F \geq 0 \ \text{ such that } \ \forall x \in \mathbb{R}_+ \quad 0 \leq F(x) \leq M_F(x+1)$$

Preliminary question : Show that any feasible plan $\underline{x_0} = (x_0, x_1, ., x_t, .) \in \Pi(x_0)$ satisfies

$$\forall t \geq 0 \quad x_0 \leq x_t \leq (2)^t x_0.$$

1. **Connection between the sequence problem and the functional equation**
For a feasible plan $\underline{x_0} = (x_0, ., x_t, .) \in \Pi(x_0)$, the cost along the plan is

$$u(\underline{x_0}) = \sum_{t=0}^{\infty} \beta^t F(2x_t - x_{t+1}).$$

The value fonction of the problem (SP) is $v^*(x_0) = \sup_{\underline{x_0} \in \Pi(x_0)} u(\underline{x_0})$.

   (a) Show that the value fonction v^* of the problem (SP) satisfies the functional equation :

$$(FE) \qquad v(x) = \sup_{x \leq y \leq 2x} \{F(2x - y) + \beta v(y)\} \qquad \forall x \in \mathbb{R}^+$$

   (b) Show that, if a feasible plan $\underline{x_0}^*$ satisfies $v^*(x_0) = u(\underline{x_0}^*)$ then

$$v^*(x_t^*) = F(2x_t^* - x_{t+1}^*) + \beta v^*(x_{t+1}^*).$$

   (c) Show that $0 \leq v^*(x_0) \leq (\frac{M_F}{1-\beta} + \frac{M_F x_0}{1-2\beta}) \leq \frac{M_F}{1-2\beta}(x_0 + 1)$

   (d) Deduce that any feasible plan $\underline{x_0}^*$ satisfying

$$v^*(x_t^*) = F(2x_t^* - x_{t+1}^*) + \beta v^*(x_{t+1}^*),$$

   is an optimal plan.

2. **Existence of a continuous solution for the functional equation**
We look for a solution of the functional equation in the following Banach space :

$$\mathcal{H}(\mathbb{R}^+) = \left\{ f : \mathbb{R}^+ \to \mathbb{R}, \text{ continuous and s. t., } \exists M \ \forall x \in \mathbb{R}^+ \ |f(x)| \leq M(|x| + 1) \right\},$$

endowed with the norme $\|f\| = \sup_{\mathbb{R}^+} \frac{|f(x)|}{|x|+1}$.
Let us define the operator T on $\mathcal{H}(\mathbb{R}^+)$ by

$$Tf(x) = \sup_{x \leq y \leq 2x} \{F(2x - y) + \beta f(y)\} \qquad \forall x \in \mathbb{R}^+$$

   (a) Show that if $f \in \mathcal{H}(\mathbb{R}^+)$ then $Tf \in \mathcal{H}(\mathbb{R}^+)$.

   (b) Show that T satisfies

      a. (monotonicity) $f, g \in \mathcal{H}(\mathbb{R}^+)$ and $f \leq g$ implies $T(f) \leq T(g)$.

      b. (discounting) for all $f \in \mathcal{H}(\mathbb{R}^+)$, $a \geq 0$.

$$T(f + a(|.| + 1)) \leq T(f) + 2\beta a|.| + \beta a \leq T(f) + 2\beta a(|.| + 1)$$

   Then deduce that T is a contraction with modulus 2β.

   (c) Show that the functional equation (FE) has a unique solution.

   (d) Show that this solution in $\mathcal{H}(\mathbb{R}^+)$ is the value function of the sequence problem.

3. **Application**
We take $F(l) = \sqrt{l}$.
Show that $v(x) = \frac{\sqrt{x}}{1-\beta}$ is a solution of the fonctional equation. Explain why this solution of the functional equation is the value function and find the optimal policy function.